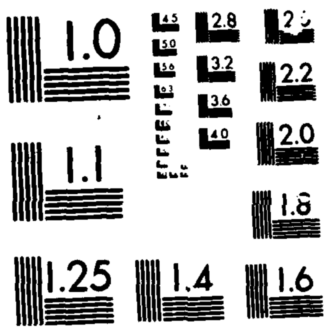
AD-A165 932 DISTANCE DETERMINATION VIA TRIANGULATION(U) 1/1
MASSACHUSETTS INST OF TECH LEXINGTON LINCOLN LAB
L G TAFF 11 FEB 86 TR-741 ESD-TR-85-303
UNCLASSIFIED F19628-85-C-0002 F/G 8/5 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

ESD-TR-86-003

Technical Report
741

Distance Determination via Triangulation

L.G. Taff

11 February 1986

Lincoln Laboratory
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LEXINGTON, MASSACHUSETTS

Prepared for the Department of the Air Force
under Electronic Systems Division Contract F19628-85-C-0002.

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the Department of the Air Force under Contract [illegible].

This report may be reproduced to satisfy needs of U.S. Government agencies.

The views and conclusions contained in this document are those of the contractor and should not be interpreted as necessarily representing the official policies, either expressed or implied, of the United States Government.

The ESD Public Affairs Office has reviewed this report, and it is releasable to the National Technical Information Service, where it will be available to the general public, including foreign nationals.

This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER

Thomas J. Alpert

Thomas J. Alpert, Major, USAF
Chief, ESD Lincoln Laboratory Project Office

Non-Lincoln Recipients

PLEASE DO NOT RETURN

Permission is given to destroy this document when it is no longer needed.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

DISTANCE DETERMINATION VIA TRIANGULATION

L.G. TAFF

Group 94

TECHNICAL REPORT 741

11 FEBRUARY 1986

LEXINGTON MASSACHUSETTS

ABSTRACT

This report investigates the feasibility of distance determination by triangulation for artificial satellites, rockets, and so forth. The two-observer case is solved analytically and investigated in detail with respect to the propagation of errors. For objects distant compared to the observer-to-observer distance $\Delta\rho$, the variance of the observer-to-object distance R is given by $\sigma_R \simeq \sigma R^2 \csc^2 \theta / \Delta\rho$. σ is the standard deviation of the angular measurements, and θ is the angle between the observer-to-observer baseline and the direction to the object. The next topic discussed is the use of multiple, simultaneous, direction determinations for triangulation. A novel method is proposed to deal with this problem. It is extended to include observations with different measurement precision, and generalized to take into account the expanding conical nature of angular errors. Finally, some data acquired at the Experimental Test System of the GEODSS network and the Millstone Hill Radar are analyzed within this context. It is clear that there now exists a powerful observational tool to aid in initial orbit determination utilizing coordinated angles-only sensors.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

DISTANCE DETERMINATION VIA TRIANGULATION

I. INTRODUCTION

The problem of distance determination by triangulation arises in many areas — surveying, passive sonar location of submarines, parallax determination in astronomy, and so forth. I have examined the subject anew to determine its utility for a space-based surveillance system which will track artificial satellites, rockets, missiles, and so on. In Section II of this report the ideal problem is solved, error estimates are provided for the results, and some complications of the real problem introduced. Section III discusses a novel method of solving the problem when more than two observers determine lines of sight. Although designed to handle multiple observers, a nice feature of this algorithm is that it reproduces the simple solution of Section II when there are only two observers.

These two early treatments assume that the statistical properties of the measurement errors of the different observers are identical. As a way of incorporating varying measurement precision, a pseudostatistical version of the problem is formulated in the fourth Section. When combined with the earlier results as a starting point and an iteration procedure designed to produce rapid convergence, this final technique has proved to be robust, easy to numerically implement, and the best of several methods tried.

The last Section returns to the two-observer case and documents the analysis of data acquired on deep-space artificial satellites at the Experimental Test System of the GEODSS network and the Millstone Hill Radar. It is clear that the theoretical expectations have been verified. Hence, a powerful, new method of passive, angles-only localization has been developed, tested, and polished for real application.

II. THE PURE PROBLEM

A. FORMULATION

An object in space has a geocentric location of $\underline{r}$. The first observer, whose geocentric location is $\underline{\rho}_1$, perceives it to be in the direction of the unit vector $\underline{\ell}_1$. Furthermore, it is an unknown distance R_1 from this observer. It follows that the topocentric location of the object is $\underline{R}_1 = R_1\underline{\ell}_1$ and that

$$\underline{R}_1 = \underline{r} - \underline{\rho}_1 \quad . \tag{1a}$$

Similarly,

$$\underline{R}_2 = R_2\underline{\ell}_2 = \underline{r} - \underline{\rho}_2 \tag{1b}$$

if the second observer measures the direction to the object at the same instant the first observer did*; see Figure 1.

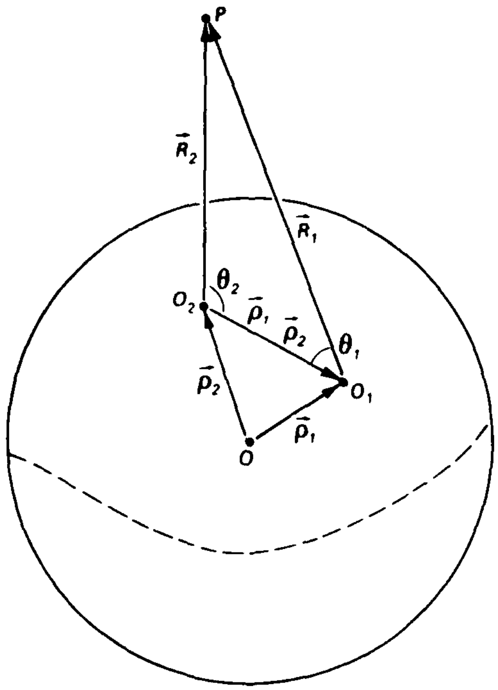


Figure 1. Geometry for simultaneous observation of an artificial satellite at P by two observers at O_1 and O_2.

124825-N-01

* Herein lies the first complication. Because light travels at a finite speed c, even if both observers are carrying synchronized clocks and are at rest relative to each other, and if they both record observations at times t_1 and t_2 and furthermore $t_1 = t_2$, they still have not made simultaneous measurements. Their data refer to locations of the object at $t_1 - R_1/c$ and $t_2 - R_2/c$, respectively. Should they be in relative motion, the whole discussion is much more complicated.

B. SOLUTION

With reference to the figure, which shows the two observers at O_1 and O_2 relative to the geocenter at O while the object is at P, let $\sphericalangle O_1O_2P = \theta_2$, $\sphericalangle O_2O_1P = \theta_1$. Then, using the definition of the scalar product,

$$\underline{R}_2 \cdot (\underline{\rho}_1 - \underline{\rho}_2) = R_2 \, |\underline{\rho}_1 - \underline{\rho}_2| \cos\theta_2$$

$$\underline{R}_1 \cdot (\underline{\rho}_2 - \underline{\rho}_1) = R_1 \, |\underline{\rho}_2 - \underline{\rho}_1| \cos\theta_1 \quad .$$

If I define $\underline{\rho}_{12}$ to be the unit vector from the first observer to the second, $\underline{\rho}_{12} = (\underline{\rho}_1 - \underline{\rho}_2)/|\underline{\rho}_1 - \underline{\rho}_2|$, then I can rewrite these two equations as

$$\cos\theta_1 = -\underline{\rho}_{12} \cdot \underline{\ell}_1$$

$$\cos\theta_2 = \underline{\rho}_{12} \cdot \underline{\ell}_2 \quad . \tag{2}$$

Since I knew where the observers were, I knew what directions they perceived the target to be in, and I know that ΔO_1PO_2 is a plane triangle (at the instant of observation), it follows that both θ_1 and θ_2 can be uniquely determined.

I can now determine R_1 and R_2. From the law of sines applied to ΔO_1PO_2 I can write

$$R_1 \csc\theta_2 = R_2 \csc\theta_1 = |\underline{\rho}_1 - \underline{\rho}_2| \csc[\pi - (\theta_1 + \theta_2)] \quad .$$

The solution for R_1 and R_2 is

$$R_1 = |\underline{\rho}_1 - \underline{\rho}_2| \sin\theta_2 \csc(\theta_1 + \theta_2)$$

$$R_2 = |\underline{\rho}_1 - \underline{\rho}_2| \sin\theta_1 \csc(\theta_1 + \theta_2) \quad . \tag{3}$$

Now that both $\underline{R}_1$ and $\underline{R}_2$ are known, $\underline{r}$ may be computed from Equation (1).

C. ANALYSIS OF VARIANCE

Inevitably there are measurement errors in $\underline{\ell}_1$ and $\underline{\ell}_2$, as well as errors in $\underline{\rho}_1$ and $\underline{\rho}_2$. They can be propagated into values for the variances of θ_1 and θ_2 and, thence, R_1 and R_2 if they are small and random. Because of the $1 \longrightarrow 2$ symmetry of the problem, I shall only discuss the variance of $R_1 = \mathrm{var}\,(R_1)$.

There are three groups of terms in the expression for the variance of θ_1. One group arises from the errors in $\underline{\rho}_1$ and $\underline{\rho}_2$ and their cross-correlations. One may reasonably expect the latter to vanish and the former to be small, equivalent to $0''.01$. As the measurement errors associated with $\underline{\ell}_1$ or $\underline{\ell}_2$ will be at least an order of magnitude larger than this, and perhaps a thousand times greater, I shall ignore all errors in the locations of the observers.

A second group of terms in the expression for var (θ_1) results from correlations between the observer's locations and the measured bearings expressed by $\underline{\ell}_1$ and $\underline{\ell}_2$. While systematic errors in the observers' locations will systematically cause $\underline{\ell}_1$ and $\underline{\ell}_2$ to be incorrect, there is no reason for

there to be any interplay between the random errors in the observers' locations and the measured positions. Hence, all these terms will be set to zero also. Of the 36 elements in the expression for var (θ_1), only three are left (21 are in the first group and 12 in the second).

The third group of terms has its origins in the errors of measurement. If, to be concrete, I write $\underline{\ell}_1$ as

$$\underline{\ell}_1 = \begin{pmatrix} \cos \Delta_1 \cos A_1 \\ \cos \Delta_1 \sin A_1 \\ \sin \Delta_1 \end{pmatrix} \quad , \quad \underline{\ell}_1 \cdot \underline{\ell}_1 = 1 \text{ by construction}$$

where Δ is a latitudinal angle and A is a longitudinal one, then the reduced expression for var (θ_1) takes the form

$$\mathrm{var}\,(\theta_1) = \left(\frac{\partial \theta_1}{\partial A_1}\right)^2 \mathrm{var}\,(A_1) + \left(\frac{\partial \theta_1}{\partial \Delta_1}\right)^2 \mathrm{var}\,(\Delta_1) + 2\left(\frac{\partial \theta_1}{\partial A_1}\right)\left(\frac{\partial \theta_1}{\partial \Delta_1}\right) \mathrm{cov}\,(A_1,\Delta_1) \quad .$$

The covariance of A_1 and Δ_1 can be made to vanish by an appropriate observing technique. Thus, the variance of θ_1 only has two significant terms.

Utilizing a similar chain of arguments, one deduces that the dominant terms in the expression for the variance of R_1 reduce to

$$\mathrm{var}\,(R_1) = \left(\frac{\partial R_1}{\partial \theta_1}\right)^2 \mathrm{var}\,(\theta_1) + \left(\frac{\partial R_1}{\partial \theta_2}\right)^2 \mathrm{var}\,(\theta_2)$$

since cov $(\theta_1,\theta_2) = 0$. From these expressions and Equations (2, 3) one can demonstrate that

$$\mathrm{var}\,(R_1) = |\underline{\rho}_1 - \underline{\rho}_2|^2 \sin^2 \theta_2 \csc^2 (\theta_1 + \theta_2) \left\{\cot^2 (\theta_1 + \theta_2)\, \mathrm{var}\,(\theta_1) + [\cot \theta_2 - \cot (\theta_1 + \theta_2)]^2\, \mathrm{var}\,(\theta_2)\right\} \quad . \tag{4}$$

Let us try and clearly see what this means. Interpret A and Δ as the usual right ascension and declination of an equatorial coordinate system. Specialize to a two-dimensional situation with Earth-based observers on the equator. Finally, let $\rho_1 = \rho_2 = \rho$, var (A_1) = var $(A_2) = \sigma^2$ (for simplicity), and set $\Delta\lambda = \lambda_2 - \lambda_1 > 0$ (i.e., the longitude difference between the observers). Now Equations (2) take the form

$$2 \cos \theta_1 = [\cos (H_1 - \Delta\lambda) - \cos H_1] \csc (\Delta\lambda/2)$$

$$2 \cos \theta_2 = [\cos (H_2 + \Delta\lambda) - \cos H_2] \csc (\Delta\lambda/2)$$

where $H = \tau - A$ is the topocentric hour angle (τ is the local sidereal time). Furthermore,

$$|\underline{\rho}_1 - \underline{\rho}_2| = 2\rho \sin (\Delta\lambda/2) \quad .$$

I next compute the variances of θ_1 and θ_2;

$$\mathrm{var}\,(\theta_1) = (\sigma/2)^2\,[\sin\,(H_1 - \Delta\lambda) - \sin H_1]^2 \csc^2\theta_1 \csc^2(\Delta\lambda/2)$$

$$\mathrm{var}\,(\theta_2) = (\sigma/2)^2\,[\sin\,(H_2 + \Delta\lambda) - \sin H_2]^2 \csc^2\theta_2 \csc^2(\Delta\lambda/2) \quad .$$

Finally,

$$\mathrm{var}\,(R_1) = \sigma^2\rho^2 \sin^2\theta_2 \csc^2(\theta_1+\theta_2)\,\Big\{\csc^2\theta_1 \cot^2(\theta_1+\theta_2) \cdot [\sin\,(H_1-\Delta\lambda) - \sin H_1]^2 + \csc^2\theta_2\,[\cot\theta_2 - \cot(\theta_1+\theta_2)]^2 \cdot [\sin\,(H_2+\Delta\lambda) - \sin H_2]^2\Big\} \quad .$$

Even this expression is formidable. As a last simplification, place the object midway between the observers. Then, $\theta_1 = \theta_2 = \theta$, $H_1 = -H_2 = H$, and $\theta + H = (\pi + \Delta\lambda)/2$. Now,

$$\mathrm{var}\,(\theta_1) = \mathrm{var}\,(\theta_2) = \sigma^2$$

$$\mathrm{var}\,(R) = (\sigma^2\rho^2/2)\sin^2\theta \sin^2(\Delta\lambda/2)\,(\csc^4\theta + \sec^4\theta) \quad .$$

This is still not transparent but becomes more so as the object-to-observer distance greatly exceeds the observer-to-observer distance. For as r, $R \to \infty$, $r \to R$ and $\theta \to \pi/2$ from below. Then, $R_1 = R_2 = R = 2|\underline{\rho}_1 - \underline{\rho}_2| \sec\theta$, $\cos\theta = (\rho/R)\sin(\Delta\lambda/2)$. Hence, writing θ as $\pi/2 - \delta\theta$ for some small $\delta\theta$,

$$\delta\theta \simeq (\rho/R)\sin(\Delta\lambda/2) \simeq (\rho/r)\sin(\Delta\lambda/2)$$

and

$$\mathrm{var}\,(R) \to \frac{\sigma^2 R^4}{2\rho^2\sin^2(\Delta\lambda/2)} \simeq \frac{\sigma^2 r^4}{2\rho^2\sin^2(\Delta\lambda/2)} \quad . \tag{5}$$

As a numerical example, consider two GEODSS sites observing a near-stationary artificial satellite. With $\Delta\lambda = 72°$ (there are to be five GEODSS installations), $r = 6.61\rho$, and $\sigma = 15''$, the standard deviation of the topocentric distance would be 17 km.

D. MORE GENERAL ANALYSIS OF VARIANCE

A generalization of Equation (5) to arbitrary geometries would be useful. I provide this now only for the case when the object is far enough away from both observers that the sum of θ_1 and θ_2 will be nearly π. Indeed, $\pi - (\theta_1 + \theta_2)$ is the small parameter of the problem.

Let $\underline{\ell}(\alpha,\delta) = (\cos\delta\cos\alpha, \cos\delta\sin\alpha, \sin\delta)$ be the unit vector of direction cosines for declination δ and right ascension α. As before, define the direction from observer 1 to be $\underline{\ell}_1 = \underline{\ell}(\alpha_1,\delta_1)$. The pointing vector from observer 2, $\underline{\ell}_2$, can be written as

$$\underline{\ell}_2 = \underline{\ell}_1 + \underline{\delta\ell}$$

for some vector $\underline{\delta\ell}$ of small norm. Moreover, because $\underline{\ell}_2$ is a unit vector, $\underline{\ell}_1 \cdot \underline{\delta\ell} \simeq 0$.

From ΔO_1O_2P in Figure 1, the relationships

$$\cos\theta_1 = -\frac{(\underline{\rho}_1 - \underline{\rho}_2)\cdot\underline{\ell}_1}{|\underline{\rho}_1 - \underline{\rho}_2|}$$

$$\cos\theta_2 = +\frac{(\underline{\rho}_1 - \underline{\rho}_2)\cdot\underline{\ell}_2}{|\underline{\rho}_1 - \underline{\rho}_2|}$$

are still valid. Setting $\theta_2 = \pi - \theta_1 - \delta\theta$ these expressions take the form

$$\cos\theta_1 = -\frac{(\underline{\rho}_1 - \underline{\rho}_2)\cdot\underline{\ell}_1}{|\underline{\rho}_1 - \underline{\rho}_2|}$$

$$\cos\theta_2 = -\cos(\theta_1 + \delta\theta) \simeq -\cos\theta_1 + \delta\theta\sin\theta_1$$

with

$$\delta\theta = \frac{(\underline{\rho}_1 - \underline{\rho}_2)\cdot\underline{\delta\ell}}{|\underline{\rho}_1 - \underline{\rho}_2|}\csc\theta_1 \quad .$$

From Equations (3), I can deduce that R_2 is given by

$$R_2 = [|\underline{\rho}_1 - \underline{\rho}_2|\sin\theta_1]^2/[(\underline{\rho}_1 - \underline{\rho}_2)\cdot\underline{\delta\ell}]$$

and

$$R_1 = R_2 + \delta R \quad , \quad \delta R = -(\underline{\rho}_1 - \underline{\rho}_2)\cdot\underline{\ell}_1 \quad .$$

As the variance of θ_1 is simply related to that of α_1,δ_1 via

$$\mathrm{var}(\theta_1) = \left(\frac{\partial\theta_1}{\partial\alpha_1}\right)^2 \mathrm{var}(\alpha_1) + \left(\frac{\partial\theta_1}{\partial\delta_1}\right)^2 \mathrm{var}(\delta_1)$$

it follows that

$$\mathrm{var}(\theta_1) = [(\underline{\rho}_{12}\cdot\partial\underline{\ell}_1/\partial\alpha_1)^2\,\mathrm{var}(\alpha_1) + (\underline{\rho}_{12}\cdot\partial\underline{\ell}_1/\partial\delta_1)^2\,\mathrm{var}(\delta_1)]\csc^2\theta_1$$

and similarly for $1 \rightarrow 2$ (where $\underline{\rho}_{12} = (\underline{\rho}_1 - \underline{\rho}_2)/|\underline{\rho}_1 - \underline{\rho}_2|$ still). Examination of the terms in the square brackets shows that, apart from the var (α_1) and the var (δ_1) terms, the remainder is some well-behaved function of the sines and cosines of α_1,δ_1 and the direction cosines of $\underline{\rho}_{12}$. Therefore, they are of order unity. Assuming, with the appropriate metric factor subsumed, that var $(\alpha_1) \simeq$ var $(\delta_1) = \sigma_1^2$, the result

$$\mathrm{var}(\theta_1) \simeq \sigma_1^2\csc^2\theta_1$$

follows. Obviously, this is just as true for the variance of θ_2 to terms of order $\delta\theta$.

The last formula needed is Equation (4). Exploiting the fact that $\theta_1 + \theta_2 = \pi - \delta\theta$, and that $\delta\theta$ is presumed to be small, the leading term is just

$$\sigma_{R_1} = [\mathrm{var}\,(R_1)]^{1/2} \simeq \frac{\sigma_1 R_1^2 \csc^2 \theta_1}{|\underline{\rho}_1 - \underline{\rho}_2|} \quad . \qquad (6)$$

The functional dependence of σ_{R_1} on R_1, $|\underline{\rho}_1 - \underline{\rho}_2|$, and θ_1 are the key elements coupled with the fact that $\csc^2 \theta_1 \geqslant 1$. The trigonometric terms buried in Equation (6) cannot play a major role, nor can the missing $\sqrt{2}$ in the expression for σ_{R_1} ($2 = 1 + 1$, one 1 for each of θ_1 and θ_2).

E. COMPLICATIONS

It seems unwise to assume that the coordination of two observers will be so successful that they will be able to make truly simultaneous observations (even with the proviso that the concept of simultaneous is not yet precisely defined). A much more realistic scenario is the following: At some time t_1 observer 1 commences a short series of observations, or a track, on the object. These independent observations are separated by some average time δt and are N_1 in number. The last one is performed at time $t_2 = t_1 + N_1\delta t$. At about time t_1 (say time T_1), observer 2 commences his track of length $N_2\delta T$, terminating at time $T_2 = T_1 + N_2\delta T$ which is near time t_2. We may even have $t_2 < T_1$ rather than $t_1 < T_1 < T_2 < t_2$, or $t_1 < T_1 < t_2 < T_2$, and so on. From these two tracks, pseudo-observations are calculated at a common epoch τ. This computation involves smoothing the two tracks in some statistical fashion, estimating the angular velocity, the angular acceleration, and probably higher order terms, and then predicting the pseudo-observations at time τ. This will be a substantially better procedure for short observation spans that appreciably overlap than for short observation spans that do not overlap, or long observation durations in general. Therefore, the σ in Equation (6) refers not to the sensor pixel size but to the standard deviation of the pseudo-observation. Very roughly, based upon the simplest least-squares track smoothing, σ will be $\simeq$ (pixel size)/$\sqrt{N}$ if the observers' locations are perfectly well-known, and so on (but see Subsection V-A). Otherwise, one should include the hardware errors and discuss an "enhanced" or "effective" pixel size which is larger than the actual pixel size.

N can not be too large, for then $N\delta t$ will be too long and the track smoothing process will be limited by the need to estimate the pseudoposition, the angular velocity, the angular acceleration, and so forth. As the $\sqrt{2}$ or $\sqrt{3}$ or even $\sqrt{4}$ improvement is not much different from unity, there may not be much gain from larger values of N at a fixed δt.

Should the observing intervals not overlap, or not even be nearly centered on τ, then the situation rapidly deteriorates. The sensitivity of the variance of the pseudopositions to $|\tau - (T_1 + T_2)/2|$ and $|\tau - (t_1 + t_2)/2|$ is *quadratic* at first. That is a very efficient way to rapidly build up errors that will swamp (pixel size)/$\sqrt{N}$. Thus, while my "realistic" scenario relaxes the perfect synchronization of the two observers, very close coordination is still necessary. I shall continue to be overly optimistic and assume this to be the case. Thus, $\sigma \simeq$ (pixel size)/2 (but see Subsection V-A).

Now it is time for the bad news. Look at Figure 2 which shows an observer making two successive measurements of a moving target. Note that, because the observer is moving, the two directions he measures are in different coordinate systems; there is planetary aberration owing to the motion of the observer. Even if you know the observer's motion you cannot correct for this. You must also know the object's distance. The majority of the effect is wiped out (via cancellation of opposing terms) if the epoch of smoothed track is the midpoint of the observing span. While I am willing to be optimistic out of ignorance, I am not so foolhardy as to believe that every pair of tracks will have the same midpoint. Therefore, in addition to accidental errors growing as $|\tau - (t_1 + t_2)/2|^2$ and $|\tau - (T_1 + T_2)/2|^2$, there will be *uncorrected systematic errors* growing quadratically with these time differences unless a very complex solid-geometry problem can be solved. So far, the two-dimensional version of it has resisted all attempts to solve it.

Is this a fine point or a major effect? For the GEODSS network it is $\simeq 300''$ (because it is composed of ground-based telescopes with a δt of 2 min). By setting τ equal to the mean epoch of the observations all but 2″ of this is taken care of. The 2″ residual represents the unmodeled acceleration of the telescope. The corresponding amount of planetary aberration for a space-based sensor platform could easily exceed a *degree*. The unmodeled part of this could reach 0°.5. As long as instantaneous, simultaneous observations represent an ideal, this source of error is inherent in this type of distance determination.

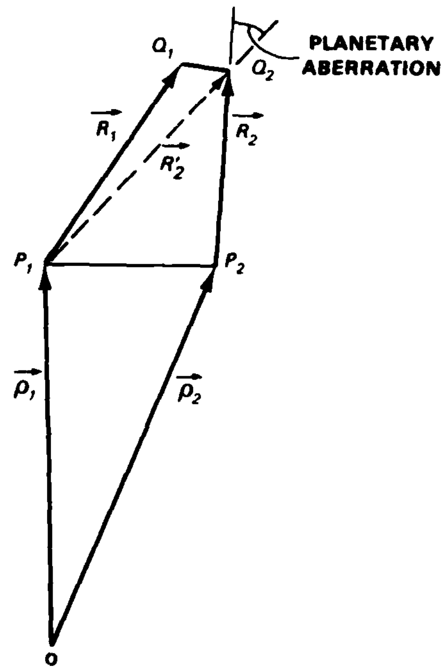


Figure 2. Illustration of planetary aberration owing to the motion of the observer ($P_1 \rightarrow P_2$) and the object of interest ($Q_1 \rightarrow Q_2$).

149706-N-03

III. THE MANY-OBSERVER PROBLEM

A. FORMULATION

The object of interest is at $\underline{r}$. There are $n = 1,2,\ldots,N$ observers located at $\underline{\rho}_1,\underline{\rho}_2,\ldots,\underline{\rho}_N$. Observer N measures the direction to the object and expresses the results as a unit vector of direction cosines $\underline{\ell}_n$. The "equations of condition" are

$$\underline{r} = \underline{\rho}_n + R_n\underline{\ell}_n \quad , \quad n = 1,2,\ldots,N$$

for the N unknown distances $R_1,R_2,\ldots,R_N$. As we saw earlier, once $N = 2$ the problem is soluble. Therefore, for $N > 3$ the problem is overdetermined and we must seek some statistical fashion to utilize our N measures of $\underline{\ell}$.

I propose to do this by exploiting the fact that the object can only occupy one place at one time. In particular, the quantity S defined by

$$S = (1/2) \sum_{\substack{n,m=1 \\ n\neq m}}^{N} |\underline{r}_m - \underline{r}_n|^2 \tag{7}$$

with $\underline{r}_m = \underline{\rho}_m + R_m\underline{\ell}_m$, will vanish for error-free $\{\underline{\rho}_m, \underline{\ell}_m\}$, and correct values of $\{R_m\}$. In practice, the errors in the observers' locations are much less than those in the bearing vectors (see above). Thus, if I regard S as a "sum of squares," then S will be least when $\underline{R} = (R_1,R_2,\ldots,R_N)$ is chosen subject to the presence of errors in $\{\underline{\ell}_n\}$.

B. SOLUTION

Since S is a quadratic function of R, if I form the "normal equations" by $\nabla_{\underline{R}} S = \underline{0}$, then I shall have a system of N linear equations to solve. In particular, this system is just

$$(N-1)R_n - \underline{\ell}_n \cdot \sum_{\substack{m=1 \\ m\neq n}}^{N} R_m\underline{\ell}_m = \underline{\ell}_n \cdot \left(\sum_{m=1}^{N} \underline{\rho}_m - N\underline{\rho}_n \right) \quad , \quad n = 1,2,\ldots,N \tag{8}$$

While the above is cast in the form of a least-squares discussion, it is not. There is no sum of squares of random errors being minimized, no unbiased estimator, and so on. Equation (7) appeals to my intuition, leads naturally to the simple Equation (8), and will be shown to have some geometrically appealing special cases (see below). Other than that, its justification must rest on its utility.

C. THE TWO-OBSERVER CASE

If $N = 2$, then S reduces to

$$S = R_1^2 + R_2^2 + \rho_1^2 + \rho_2^2 + 2R_1\underline{\ell}_1 \cdot \underline{\rho}_1 + 2R_2\underline{\ell}_2 \cdot \underline{\rho}_2$$
$$- 2(R_1R_2\underline{\ell}_1 \cdot \underline{\ell}_2 + R_1\underline{\ell}_1 \cdot \underline{\rho}_2 + R_2\underline{\ell}_2 \cdot \underline{\rho}_1 + \underline{\rho}_1 \cdot \underline{\rho}_2) \quad .$$

From $\partial S/\partial R_1 = 0$ and $\partial S/\partial R_2 = 0$, one derives ($\cos\theta_{12} = \underline{\ell}_1 \cdot \underline{\ell}_2$)

$$R_1 - R_2 \cos\theta_{12} = -|\underline{\rho}_1 - \underline{\rho}_2|\underline{\rho}_{12} \cdot \underline{\ell}_1$$

$$-R_1 \cos\theta_{12} + R_2 = |\underline{\rho}_1 - \underline{\rho}_2|\underline{\rho}_{12} \cdot \underline{\ell}_2 \quad .$$

The determinant of the system is just $\sin^2\theta_{12}$. Hence, there cannot be a solution if $\theta_{12} = 0$ or π (e.g., $\underline{\ell}_1$ and $\underline{\ell}_2$ parallel or anti-parallel). This should be geometrically obvious.

The solution has the form

$$R_1 = -(\underline{\ell}_1 - \underline{\ell}_2 \cos\theta_{12}) \cdot \underline{\rho}_{12}|\underline{\rho}_1 - \underline{\rho}_2| \csc^2\theta_{12}$$

$$R_2 = (\underline{\ell}_2 - \underline{\ell}_1 \cos\theta_{12}) \cdot \underline{\rho}_{12}|\underline{\rho}_1 - \underline{\rho}_2| \csc^2\theta_{12} \quad .$$

Note that $\underline{\ell}_1 - \underline{\ell}_2 \cos\theta_{12}$ is perpendicular to $\underline{\ell}_2$, $\underline{\ell}_2 - \underline{\ell}_1 \cos\theta_{12}$ is orthogonal to $\underline{\ell}_1$, and that these are merely Equations (3) in a different notation [$\theta_{12} = \pi - \theta_1 - \theta_2$ and use Equations (2)]. The nonexistence of a solution in the intuitively obvious cases and the reduction to the deterministic solution instill confidence in the meaningfulness of expression (7).

D. THE THREE-OBSERVER CASE

While in this vein, note that if N = 1, then Equation (8) just reads 0 = 0. More interesting is the determinant of Equation (8) for N = 3. With $\cos\theta_{nm} = \underline{\ell}_n \cdot \underline{\ell}_m$, it may be written as

$$8 - 2\cos\theta_{12}\cos\theta_{13}\cos\theta_{23} - 2(\cos^2\theta_{13} + \cos^2\theta_{23} + \cos^2\theta_{12}) \quad .$$

Once again, this prohibits a solution in the geometrically clear *instances* — two θ_{nm} equal to π, and the third one zero.

IV. THE FINAL ALGORITHM

A. MOTIVATION

Once again consider an observer located at ρ who perceives the object in the direction $\underline{\ell}$. The likelihood function for the object's location $\underline{r}$ is

$$f(\underline{r}) = C(ww')^{1/2} \exp\left[-(\underline{r} - \underline{\rho})^T M(\underline{r} - \underline{\rho})/2\right] \tag{9}$$

where superscript T denotes transpose and the covariance matrix M is given by

$$M = w\underline{m}\underline{m}^T + w'\underline{n}\underline{n}^T$$

C is a normalization constant, while w and w′ are weights that provide information concerning the precision of the observations along the directions m and n. These two unit vectors are orthogonal to $\underline{\ell}$. If the measurement errors are symmetric in the plane perpendicular to $\underline{\ell}$, then M has the form of a projection operator, viz.

$$M = w(\underline{m}\underline{m}^T + \underline{n}\underline{n}^T) = w(I - \underline{\ell}\underline{\ell}^T) \quad .$$

The likelihood function has its maximum along the line whose tangent is $\underline{\ell}$ and is constant on all parallel lines. Furthermore, C is not truly a normalization constant, for $\int f(\underline{r})d\underline{r}$ does not exist and the expected value of $\underline{r}$ is not defined. If I adapt this model to the two-observer case then these difficulties disappear. Now, $f(\underline{r})$ is given by

$$f(\underline{r}) = Cf_1(\underline{r})\, f_2(\underline{r})$$

where each likelihood function has the form in Equation (9). The mean value for $\underline{r}$ is

$$<\underline{r}> = (M_1 + M_2)^{-1} (M_1\underline{\rho}_1 + M_2\underline{\rho}_2) \quad .$$

Its covariance matrix is just $(M_1 + M_2)^{-1}$. This matrix exists if and only if the two observers plus object are not collinear.

For N > 2 observers, the process is simply generalized. Each observer has his own location ($\underline{\rho}_n$), his own vector of direction cosines ($\underline{\ell}_n$), covariance matrix (M_n), weights (w_n,w'_n), and so forth. The overall likelihood function is given by the product of the individual likelihood functions. It is maximized for

$$\underline{r} = <\underline{r}> \equiv \left(\sum_{n=1}^{N} M_n\right)^{-1} \left(\sum_{m=1}^{N} M_m\underline{\rho}_m\right) \tag{10}$$

whose covariance matrix is

$$\left(\sum_{n=1}^{N} M_n\right)^{-1} \quad .$$

Where do the weights w,w′ come from? Ordinarily, weights are inversely proportional to the squares of the standard deviations of the measurement errors. In this problem, that would not take into account the fact that a fixed angular error corresponds to different linear errors at different distances. Hence (see Figure 3 for motivation), the proposal is made that for circularly symmetric measurement errors,

$$w = w' = 1/(R\sigma)^2 \quad .$$

The notation is as above; σ is the angular measurement error of an observer, and R is the topocentric distance of the object in question. Of course, the R_n are unknown *a priori.*

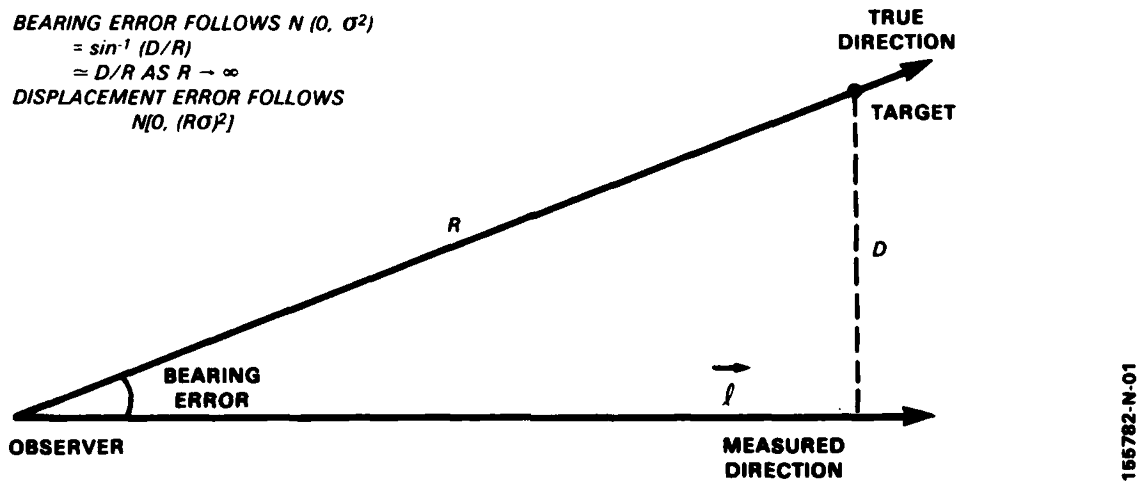


Figure 3. Motivation for $w = 1/(R\sigma)^2$. If the bearing errors are normally distributed with zero mean and standard deviation σ, i.e., as $N(0,\sigma^2)$, then the displacement errors in the tangential direction are distributed as $N[0,(R\sigma)^2]$ for small σ.

B. METHOD

The above description seems a reasonable way to proceed in allowing for asymmetric measurement errors, various numbers of observers with different measurement precisions, and the incompleteness of the angles-only information. However, without the distance information the weights can not be computed. Therefore, I recommend the following procedure:

(1) Solve for r by using the method of Section III. Here, r is understood to be

$$\sum_{n=1}^{N} (R_n \underline{\ell}_n + \underline{\rho}_n)/N \quad .$$

(2) Use the set of topocentric distances and the known measurement errors of the observers to compute the set of weights w_n, w'_n ; viz.

$$w_n = 1/(R_n \sigma_n)^2 \quad , \quad w'_n = 1/(R_n \sigma'_n)^2 \quad .$$

If the measurement errors are symmetric, then $\sigma_n = \sigma'_n$.

(3) Compute, with these weights, the covariance matrices $\{M_n\}$ and then the solution for $\underline{r}$ from Equation (10).

(4) Calculate new values of the topocentric distances from

$$R_n = \underline{\ell}_n \cdot (\underline{r} - \underline{\rho}_n) \quad .$$

If the difference between these topocentric distances and the old set is sufficiently small, then terminate. Otherwise, return to step (2).

This method is straightforward, numerically simple, robust, has an obvious convergence criterion, and worked better than several other algorithms tested. (The details of the numerical testing and of the competing algorithms are being prepared for later publication elsewhere.)

V. EXPERIMENTAL TESTS

The Millstone Hill Radar is equipped with a co-mounted 14-in. telescope. Having a wide field of view, the use of the telescope in conjunction with the acquisition of an artificial satellite speeds the radar on its mission. Unfortunately, no simple means exist to accurately and independently calibrate the telescope. Hence, I used directional data acquired by the radar, rounded to the nearest 0°01, as pseudo-optical data. This precision is only slightly worse than that obtainable at the ETS in its Single Star Calibration mode. On the night of 2 January 1984, with telephone contact between the ETS and the Millstone Hill Radar, a short series of tracks were nearly simultaneously acquired on six deep-space satellites. A total of eight sets of tracks were obtained with four to nine observations per track. The observing frequency was once every 30 to 60 s. These data were smoothed by separate least-squares fits to second-order polynomials and a pseudo-observation generated at a common epoch. These pseudo-observations were $\underline{\ell}_1$ and $\underline{\ell}_2$ of Equations (2). The common epoch was the mean of the average times of observation. This can be shown to be the best choice when the two sets of observations are equally and symmetrically spaced about their individual midpoints. Next, knowing the locations of the two sensors, one can compute $\underline{\rho}_1$, $\underline{\rho}_2$, $\underline{\rho}_{12}$, and thence θ_1 and θ_2. Finally, R_1 and R_2 were computed from Equations (3).

Having calculated R_1 and R_2, I want to do at least two things: assess their reliability against the "correct" answer, and compare the magnitude of the deviations with the theoretical analysis of variance presented in Section II-C. What is the "correct" answer? An obvious choice for R_1 is the distance actually measured by the radar. Such a comparison is shown in the third column of Table I (the units are kilometers). Actually, the measured distances have been smoothed with a quadratic polynomial in the time and a pseudodistance generated from the parameters of the fit at the pseudo-observation time. Since the residuals from the least-squares fit of the raw distances are generally much less than the distance mismatches (see the last column of Table I), this artifice seems harmless.

A second choice for "correct" might be the distance predicted from the orbital element sets of these known artificial satellites at the pseudo-observation instant. I used Space Defense Center element sets and ETS software to predict R_l values. The values of the differences between these topocentric distances and the deduced ones are in the fourth column of Table I. There is a very high degree of correlation between the values in the third and fourth columns of the Table. Since we may expect that the Space Defense Center gives high weight to Millstone Hill observations, and that the ETS software is correct, the correlation between these values is not a surprising development. Presumably, had I utilized Millstone Hill software and element sets, the correspondence would have been even tighter (and more circular in its logic).

Can one predict the magnitude of these discrepancies from the analysis of variance given earlier? In general, yes. For the standard deviation of an individual right ascension or declination (actually, horizon system coordinates are returned by the radar) one can not use the canonically quoted values of 15″ for the ETS and 18″ for Millstone Hill. (However, predictions of the variance of R_1 utilizing these values are given in the fifth column of Table I for comparison. Note

TABLE I Values of R_1 and Its Errors*						
Satellite Track	R_1 Calculation	R_1-MH Value	R_1-ElSet Value	Standard Deviations of R_1		
				Theoretical		From Fit
				15″	Actual	
1	15580.5	71.1	31.5	9.6	2243.5	8.7
2	25607.6	-43.2	15.2	15.6	1515.4	3.6
3	8817.8	60.9	19.5	2.2	785.9	2.6
4	26591.1	219.2	188.8	16.8	3008.2	11.7
5	40761.4	895.7	901.9	39.5	3271.5	6.2
6	38090.9	323.1	310.0	47.4	10649.2	33.7
7	7659.5	-21.9	-71.8	1.3	1236.8	2.8
8	18364.6	-659.7	-577.7	7.9	1495.9	10.4

* All quantities are in kilometers.

that they grossly underestimate the actual differences.) A better value for the standard deviation of an altitude a or azimuth A is the residual from the least-squares polynomial smoothing. Better, but not best. I need a slight statistical diversion to further elucidate this point. However, before delving into the statistics of the problem, observe that the sixth column of Table I does contain predicted standard deviations utilizing these values. Moreover, the magnitudes are generally in consonance with the third or fourth columns of the Table.

A. STATISTICAL DIVERSION

The altitude and azimuth (a,A) generated at the pseudo-observation time are the result of a statistical adjustment of the observed altitudes and azimuths $\{a_n,A_n\}$ and their times of observations $\{t_n,t'_n\}$, n = 1,2,...,N. This explicitly includes the computation of estimates of the parameters $p_1,p_2,...,p_M$ of some particular model fit (a polynomial in time which is linear in the p's). A result of this process can be the error covariance matrix P of the parameters.

The altitude (or azimuth) at the pseudo-observation time can be written as

$$a = a(a_1,a_2,...,a_N;\ p_1,p_2,...,p_M;\ t_1,t_2,...,t_N;\ t'_1,t'_2,...,t'_N) \quad .$$

The predicted Millstone altitude depends not only on the observed Millstone variables, the

parameters $\underline{p} = (p_1,p_2,...,p_M)$, the observation times $(t_1,t_2,...,t_N)$, but also on the times of observation of the same satellite at the ETS $(t'_1,t'_2,...,t'_N)$ for the pseudo-observation instant depends on all the times. Fortunately, the errors associated with the times of observation are so small I can ignore them in the ensuing discussion. Thus, all times are considered to be exact and the explicit functional dependence of a or A on them will not be exhibited.

Now, there are errors of observation associated with the $\{a_n, A_n\}$. Let me symbolize this by a square array of dimension N whose main diagonal contains the variances of the measurements. Call it V. Therefore, the covariance matrix of the arguments of the predicted altitude takes the form

$$C = \begin{pmatrix} V & O \\ O & P \end{pmatrix} .$$

The covariance matrix of a and A has the form of a product,

$$\begin{pmatrix} \left(\frac{\partial a}{\partial \underline{d}}\right)^T & \left(\frac{\partial a}{\partial \underline{p}}\right)^T \\ \left(\frac{\partial A}{\partial \underline{d}}\right)^T & \left(\frac{\partial A}{\partial \underline{p}}\right)^T \end{pmatrix} \quad C \quad \begin{pmatrix} \left(\frac{\partial a}{\partial \underline{d}}\right) & \left(\frac{\partial A}{\partial \underline{d}}\right) \\ \left(\frac{\partial a}{\partial \underline{p}}\right) & \left(\frac{\partial A}{\partial \underline{p}}\right) \end{pmatrix}$$

wherein $\underline{d} = (a_1,a_2,...,a_N;\ A_1,A_2,...,A_N)$ is a vector of data (in practice it simplifies matters somewhat that $\underline{d}$ splits into two independent parts).

If I write out the above matrix products, then I obtain

$$\begin{pmatrix} \left(\frac{\partial a}{\partial \underline{d}}\right)^T V \left(\frac{\partial a}{\partial \underline{d}}\right) + \left(\frac{\partial a}{\partial \underline{p}}\right)^T P \left(\frac{\partial a}{\partial \underline{p}}\right) & \left(\frac{\partial a}{\partial \underline{d}}\right)^T V \left(\frac{\partial A}{\partial \underline{d}}\right) + \left(\frac{\partial a}{\partial \underline{p}}\right)^T P \left(\frac{\partial A}{\partial \underline{p}}\right) \\ \left(\frac{\partial A}{\partial \underline{d}}\right)^T V \left(\frac{\partial a}{\partial \underline{d}}\right) + \left(\frac{\partial A}{\partial \underline{p}}\right)^T P \left(\frac{\partial a}{\partial \underline{p}}\right) & \left(\frac{\partial A}{\partial \underline{d}}\right)^T V \left(\frac{\partial A}{\partial \underline{d}}\right) + \left(\frac{\partial A}{\partial \underline{p}}\right)^T P \left(\frac{\partial A}{\partial \underline{p}}\right) \end{pmatrix}$$

Examine, for example, the variance of the azimuth A. It takes the form

$$\sigma_A^2 = \sum_{n=1}^{2N} \left(\frac{\partial A}{\partial d_n}\right)^2 \sigma_n^2 + \left(\frac{\partial A}{\partial \underline{p}}\right)^T P \left(\frac{\partial A}{\partial \underline{p}}\right) .$$

The first term represents the contribution to the variance of A from the random errors in the measured altitudes $(d_1 = a_1, d_2 = a_2, ..., d_N = a_N)$ and azimuths $(d_{N+1} = A_1, d_{N+2} = A_2, ..., d_{2N} = A_N)$. The curvilinear nature of the horizon coordinate system implies that the altitude errors have no bearing in practice. The variance of the n^{th} element of d_n is σ_n^2. Also note that each element of d contributes individually, and the aggregrate is a sum of squares.

In contrast, look at the second term in σ_A^2. It involves contributions from all the parameters of the fit. Moreover, the derivatives of a or A with respect to $\underline{p}$ may themselves be functions of a or A. Thus, the "accidental" errors in the fitting functions create systematic errors in a and A. The residuals from the fit contain contributions from such sources. This explains why the values

for the variance of R_1 as computed using these values are so much larger than those obtained using the "nominal" position uncertainties. These residuals could also be reduced by utilizing a more realistic model to fit the data to. (The answer is not merely a higher-order polynomial. All the computations described in this Section were also performed using a cubic fit — with marginal improvement.) The purpose of this digression is to explicitly demonstrate why the simple "pixel size/$\sqrt{N}$" value will prove to be a significant underestimate of the true angular uncertainty.

ACKNOWLEDGMENTS

Computational support was provided by Patrice M.S. Randall and D.M. Jonuskis. The development of the iterative method in Section IV was by B.G. Morton and H.R. Richardson of D.H. Wagner Associates, under my direction. The deciphering of my handwriting is the accomplishment of Lori McGonagle.

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ESD-TR-85-303	2. GOVT ACCESSION NO. ADA 165932	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Distance Determination via Triangulation		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER Technical Report 741
7. AUTHOR(s) Laurence G. Taff		8. CONTRACT OR GRANT NUMBER(s) F19628-85-C-0002
9. PERFORMING ORGANIZATION NAME AND ADDRESS Lincoln Laboratory, M.I.T. P.O. Box 73 Lexington, MA 02173-0073		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Program Element No. 63220C
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Systems Command, USAF Andrews AFB Washington, DC 20334		12. REPORT DATE 11 February 1986
		13. NUMBER OF PAGES 32
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Electronic Systems Division Hanscom AFB, MA 01731		15. SECURITY CLASS. (of this Report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

None

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

parallax	initial orbit determination
satellite orbits	angles-only distance determination

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report investigates the feasibility of distance determination by triangulation for artificial satellites, rockets, and so forth. The two-observer case is solved analytically and investigated in detail with respect to the propagation of errors. For objects distant compared to the observer-to-observer distance $\Delta\rho$, the variance of the observer-to-object distance R is given by $\sigma_R \simeq \sigma R^2 \csc^2 \theta/\Delta\rho$. σ is the standard deviation of the angular measurements, and θ is the angle between the observer-to-observer baseline and the direction to the object. The next topic discussed is the use of multiple, simultaneous, direction determinations for triangulation. A novel method is proposed to deal with this problem. It is extended to include observations with different measurement precision, and generalized to take into account the expanding conical nature of angular errors. Finally, some data acquired at the Experimental Test System of the GEODSS network and the Millstone Hill Radar are analyzed within this context. It is clear that there now exists a powerful observational tool to aid in initial orbit determination utilizing coordinated angles-only sensors.

END

FILMED

4-86

DTIC